

BULK GREEN'S FUNCTIONS IN ONE-DIMENSIONAL UNSTEADY PROBLEMS OF ELASTIC DIFFUSION

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Abstract. We consider a one-dimensional unsteady problem of elasticity with diffusion and preset unsteady volumetric disturbances. The mathematical model is based on a local equilibrium model of elastic diffusion. The solution is sought in integral form. The bulk Green's functions are found via Laplace transform and Fourier transform for unbounded medium, sine and cosine transform for semi-bounded medium, Fourier's series for bounded medium.

Keywords: elastic diffusion, unsteady problems, Green's functions, integral transformations

1. Introduction

In a number of cases, to calculate the stress-strain state of the medium, it is necessary to take into account the action of mass forces (weight, temperature field, etc.) or some initial spatial distribution of physical fields. The solution of such problems is expressed in terms of the bulk Green's functions.

The problems of elastic diffusion were considered in [1-9]. As a rule, these problems are solved in a stationary formulation [6]. The solution of unsteady problems is sought using the Laplace transform. The inversion of the Laplace transform is performed numerically using the Durbin algorithm and its modifications [7-9]. Publications devoted to the construction of Green's functions of unsteady problems for elastic diffusion are not known at present.

The algorithm of finding the bulk Green's functions for a one-dimensional unsteady problem of elastic diffusion is considered.

2. Statement of the problem

Let us consider a homogeneous N -component solid continuum affected by volumetric unsteady elastic diffusion disturbances.

Physical and mechanical processes in a rectangular Cartesian coordinate system are described by coupled equations of motion and mass transfer (the point is time derivative by τ , the prime is coordinate derivatives by x) [5-13]:

$$\begin{aligned} \ddot{u}(x, \tau) &= u''(x, \tau) - \sum_{j=1}^N \alpha_j \eta_j'(x, \tau) + F_1(x, \tau), \\ \dot{\eta}_q(x, \tau) &= D_q \eta_q''(x, \tau) - \Lambda_q u'''(x, \tau) + F_{q+1}(x, \tau) \quad (q = \overline{1, N}). \end{aligned} \quad (1)$$

All values here are dimensionless. Their connection with dimensional analogues will be expressed as follows:

$$x = \frac{x_1}{L}, u = \frac{u_1}{L}, \tau = \frac{Ct}{L}, \eta_q = \frac{\eta^{(q)}}{n_0}, C^2 = \frac{\lambda + 2\mu}{\rho}, F_1 = \frac{\rho LF}{\lambda + 2\mu}, F_{q+1} = \frac{LF^{(q)}}{C},$$

$$n_0 = \sum_{q=1}^N n_0^{(q)}, \alpha_q = \frac{\alpha^{(q)}}{\lambda + 2\mu}, D_q = \frac{D^{(q)}}{CL}, \Lambda_q = \frac{\Lambda^{(q)}}{CL}, \Lambda^{(q)} = \frac{n_0^{(q)} m^{(q)} D^{(q)} \alpha^{(q)}}{\rho RT_0},$$

where x_i is Cartesian coordinate (Ox_1 axis is normal to the boundary of the domain); t is time; u_i is displacement; T_0 is initial temperature; $\eta^{(q)} = n^{(q)} - n_0^{(q)}$ is concentration increment; $n^{(q)}$ and $n_0^{(q)}$ is actual and initial concentration of component q ; $\alpha^{(q)}$ is coefficient of volume expansion due to mass transfer; R is the universal gas constant; $D^{(q)}$ is coefficient of self-diffusion; λ, μ are the elastic Lamé constants; ρ is density of the medium, $m^{(q)}$ is molar mass; F and $F^{(q)}$ are the bulk perturbations; L is the layer thickness or is the characteristic length of half-space.

Then we consider initial boundary value problems for the following domains:

- unbounded domain (Cauchy problem) $-\infty < x < \infty$;

- semi-bounded problem (one-dimensional half-space) $x > 0$, with boundary conditions as follows:

$$u|_{x=0} = 0, J_q|_{x=0} = 0, u, \eta_q, \sigma, J_q = O(1) \quad (x \rightarrow \infty), \quad q = \overline{1, N}, \quad (2)$$

or

$$\sigma|_{x=0} = 0, \eta_q|_{x=0} = 0, u, \eta_q, \sigma, J_q = O(1) \quad (x \rightarrow \infty), \quad q = \overline{1, N}. \quad (3)$$

- bounded domain (one-dimensional layer) $0 < x < 1$ with the boundary conditions:

$$u|_{x=0} = 0, J_q|_{x=0} = 0, u|_{x=1} = 0, J_q|_{x=1} = 0, \quad q = \overline{1, N}, \quad (4)$$

or

$$\sigma|_{x=0} = 0, \eta_q|_{x=0} = 0, \sigma|_{x=1} = 0, \eta_q|_{x=1} = 0, \quad (5)$$

or

$$u|_{x=0} = 0, J_q|_{x=0} = 0, \sigma|_{x=1} = 0, \eta_q|_{x=1} = 0, \quad (6)$$

with initial conditions for each problems assumed to be zero

$$u_i|_{\tau=0} = \dot{u}_i|_{\tau=0} = \eta_q|_{\tau=0} = 0. \quad (7)$$

Here J_q is density of diffusion flow for component q , $\sigma = \sigma_{11}$ is stress tensor component, which is found by the formula [5-13]:

$$J_q = \Lambda_q u'' - D_q \eta'_q, \quad \sigma = u' - \sum_{j=1}^N \alpha_j \eta_j.$$

3. Solution method

Let us introduce functions $G_{km}(x, \xi, \tau)$, which will be understood as follows: $G_{km}^h(x, \xi, \tau)$ - bulk Green's functions for half-space, $G_{km}^l(x, \xi, \tau)$ - bulk Green's functions for a layer.

In accordance with (1) they satisfy the system of equations ($q = \overline{1, N}, m = \overline{1, N+1}$):

$$\ddot{G}_{1m} = G_{1m}'' - \sum_{j=1}^N \alpha_j G'_{j+1,m} + \delta_{1m} \Delta_1, \dot{G}_{q+1,m} = D_q G''_{q+1,m} - \Lambda_q G'''_{1m} + \delta_{q+1,m} \Delta_{q+1}, \quad (8)$$

$$\Delta_m = \delta(x - \xi) \delta(\tau),$$

and homogeneous boundary conditions (2) or (3) for functions G_{km}^h , (4) or (5) or (6) for functions G_{km}^l .

Here δ_{1m} is the Kronecker symbol, $\delta(\tau)$ is the Dirac delta function, the prime is coordinate derivatives by x . The initial condition of all functions are represented as (7).

A general solution of the problem (1) with corresponding homogeneous boundary conditions will take the form:

$$u(x, \tau) = \sum_{k=1}^{N+1} \int_0^\tau \int_0^Z G_{1k}(x, \xi, \tau - t) F_k(\xi, t) d\xi dt, \quad \eta_q(x, \tau) = \sum_{k=1}^{N+1} \int_0^\tau \int_0^Z G_{q+1,k}(x, \xi, \tau - t) F_k(\xi, t) d\xi dt. \quad (9)$$

where $Z = +\infty$ in problems for half-space, $Z = 1$ in problems for a layer.

Let us find functions G_{km}^l . For this, purpose we will apply the Laplace transform and Fourier series to the system (8). This will result into (here small upper indices « s » and « c » stand for sine and cosine harmonics of Fourier series, upper index « L » is Laplace's transform, « s » is parameter of Laplace transform, $\lambda_n = \pi n$):

- for boundary conditions (4)

$$G_{q+1,m}^{LLc}(0, \xi, s) = s^{-1} \delta_{q+1,m}; \quad (10)$$

$$(\lambda_n^2 + s^2) G_{1m}^{LLs}(\lambda_n, \xi, s) - \sum_{j=1}^N \alpha_j \lambda_n G_{j+1,m}^{LLc}(\lambda_n, \xi, s) = 2\delta_{1m} \cos \lambda_n \xi, \quad (11)$$

$$-\Lambda_q \lambda_n^3 G_{1m}^{LLs}(\lambda_n, \xi, s) + (s + D_q \lambda_n^2) G_{q+1,m}^{LLc}(\lambda_n, \xi, s) = 2\delta_{q+1,m} \sin \lambda_n \xi;$$

$$G_{1m}^{LL}(x, \xi, s) = \sum_{n=1}^{\infty} G_{1m}^{LLs}(\lambda_n, \xi, s) \sin \lambda_n x, \quad G_{q+1,m}^{LL}(x, \xi, s) = \sum_{n=0}^{\infty} G_{q+1,m}^{LLc}(\lambda_n, \xi, s) \cos \lambda_n x, \quad (12)$$

$$G_{1m}^{LLs}(\lambda_n, \xi, s) = 2 \int_0^1 G_{1m}^{LL}(x, \xi, s) \sin \lambda_n x dx, \quad G_{q+1,m}^{LLc}(\lambda_n, \xi, s) = 2 \int_0^1 G_{q+1,m}^{LL}(x, \xi, s) \cos \lambda_n x dx.$$

- for boundary conditions (5)

$$G_{1m}^{LLc}(0, \xi, s) = s^{-2} \delta_{1m}; \quad (13)$$

$$(\lambda_n^2 + s^2) G_{1m}^{LLc}(\lambda_n, \xi, s) + \sum_{j=1}^N \alpha_j \lambda_n G_{j+1,m}^{LLs}(\lambda_n, \xi, s) = 2\delta_{1m} \sin \lambda_n \xi, \quad (14)$$

$$\Lambda_q \lambda_n^3 G_{1m}^{LLc}(\lambda_n, \xi, s) + (s + D_q \lambda_n^2) G_{q+1,m}^{LLs}(\lambda_n, \xi, s) = 2\delta_{q+1,m} \cos \lambda_n \xi;$$

$$G_{1m}^{LL}(x, \xi, s) = \sum_{n=0}^{\infty} G_{1m}^{LLc}(\lambda_n, \xi, s) \cos \lambda_n x, \quad G_{q+1,m}^{LL}(x, \xi, s) = \sum_{n=1}^{\infty} G_{q+1,m}^{LLs}(\lambda_n, \xi, s) \sin \lambda_n x, \quad (15)$$

$$G_{1m}^{LLc}(\lambda_n, \xi, s) = 2 \int_0^1 G_{1m}^{LL}(x, \xi, s) \cos \lambda_n x dx, \quad G_{q+1,m}^{LLs}(\lambda_n, \xi, s) = 2 \int_0^1 G_{q+1,m}^{LL}(x, \xi, s) \sin \lambda_n x dx.$$

The solutions of the system (11) are written as follows:

$$\left\{ \begin{array}{l} G_{11}^{lls}(\lambda_n, \xi, s) \\ G_{1,q+1}^{lls}(\lambda_n, \xi, s) \end{array} \right\} = 2 \left\{ \begin{array}{l} P_{11}(\lambda_n, s) \sin \lambda_n \xi \\ P_{1,q+1}(\lambda_n, s) \cos \xi \lambda_n \end{array} \right\} \frac{1}{P(\lambda_n, s)} \quad (p, q = \overline{1, N}),$$

$$G_{q+1,1}^{llc}(\lambda_n, \xi, s) = 2 \frac{P_{q+1,1}(\lambda_n, s)}{P(\lambda_n, s)} \sin \lambda_n \xi, \quad (16)$$

$$G_{q+1,p+1}^{llc}(\lambda_n, \xi, s) = 2 \left[\frac{\delta_{pq}}{s + D_q \lambda_n^2} + \frac{P_{q+1,p+1}(\lambda_n, s)}{Q_q(\lambda_n, s)} \right] \cos \lambda_n \xi,$$

$$P_{1,q+1}(\lambda_n, s) = \alpha_q \lambda_n \prod_{r \neq q}^N (s + D_r \lambda_n^2), \quad P_{q+1,p+1}(\lambda_n, s) = \alpha_p \Lambda_q \lambda_n^4 \prod_{r \neq p}^N (s + D_r \lambda_n^2),$$

$$P_{11}(\lambda_n, s) = \prod_{j=1}^N (s + D_j \lambda_n^2), \quad P_{q+1,1}(\lambda_n, s) = \lambda_n^3 \Lambda_q \prod_{j=1, j \neq q}^N (s + D_j \lambda_n^2) \quad (p, q = \overline{1, N}).$$

Similarly, the solutions of the system (14) are written as follows:

$$G_{q+1,1}^{lls}(\lambda_n, \xi, s) = 2 \frac{P_{q+1,1}(\lambda_n, s)}{P(\lambda_n, s)} \cos \xi \lambda_n,$$

$$\left\{ \begin{array}{l} G_{11}^{llc}(\lambda_n, \xi, s) \\ G_{1,q+1}^{llc}(\lambda_n, \xi, s) \end{array} \right\} = 2 \left\{ \begin{array}{l} P_{11}(\lambda_n, s) \cos \xi \lambda_n \\ P_{1,q+1}(\lambda_n, s) \sin \xi \lambda_n \end{array} \right\} \frac{1}{P(\lambda_n, s)}, \quad (17)$$

$$G_{q+1,p+1}^{lls}(\lambda_n, \xi, s) = 2 \left[\frac{\delta_{pq}}{s + D_q \lambda_n^2} + \frac{P_{q+1,p+1}(\lambda_n, s)}{Q_q(\lambda_n, s)} \right] \sin \xi \lambda_n,$$

$$P_{1,q+1}(\lambda_n, s) = -\alpha_q \lambda_n \prod_{r \neq q}^N (s + D_r \lambda_n^2), \quad P_{q+1,p+1}(\lambda_n, s) = \alpha_p \Lambda_q \lambda_n^4 \prod_{r \neq p}^N (s + D_r \lambda_n^2),$$

$$P_{11}(\lambda_n, s) = \prod_{j=1}^N (s + D_j \lambda_n^2), \quad P_{q+1,1}(\lambda_n, s) = -\lambda_n^3 \Lambda_q \prod_{j=1, j \neq q}^N (s + D_j \lambda_n^2) \quad (p, q = \overline{1, N}).$$

The polynomials P and Q_q in the equations (16) and (17) are determined by the formulas:

$$P(\lambda_n, s) = (s^2 + \lambda_n^2) \prod_{j=1}^N (s + D_j \lambda_n^2) - \sum_{j=1}^N \alpha_j \Lambda_j \lambda_n^4 \prod_{r=1, r \neq j}^N (s + D_r \lambda_n^2),$$

$$Q_q(\lambda_n, s) = P(\lambda_n, s) (s + D_q \lambda_n^2).$$

Note. The Green's functions for the layer and for the half-space are related as follows (λ is a parameter of sine and cosine transform, upper index «C» is cosine transform, upper index «S» is a sine transform)

$$\left\{ \begin{array}{l} G_{km}^{hLC}(\lambda, \xi, s) \\ G_{km}^{hLS}(\lambda, \xi, s) \end{array} \right\} = \left\{ \begin{array}{l} G_{km}^{llc}(\lambda, \xi, s) \\ G_{km}^{lls}(\lambda, \xi, s) \end{array} \right\}, \quad G_{km}^{hL}(x, \xi, s) = \frac{1}{\pi} \int_0^\infty \left\{ \begin{array}{l} G_{km}^{hLS}(\lambda, \xi, s) \sin \lambda x \\ G_{km}^{hLC}(\lambda, \xi, s) \cos \lambda x \end{array} \right\} d\lambda.$$

The Laplace transition to domain of originals in the equations (10), (12), (13), (15) - (16) is reduced to inversion of the following expressions (v is any of parameters λ or λ_n):

$$\frac{1}{s}, \quad \frac{1}{s^2}, \quad \frac{1}{s + D_q v^2}, \quad \frac{P_m(v, s)}{P(v, s)}, \quad \frac{P_{q+1,m}(v, s)}{Q_q(v, s)}.$$

Their originals are found through deductions and on operational calculus tables and [12,13]:

$$L^{-1}\left[\frac{1}{s}\right] = H(\tau), \quad L^{-1}\left[\frac{1}{s^2}\right] = \tau, \quad L^{-1}\left[\frac{1}{s + D_q v^2}\right] = \exp(-D_q v^2 \tau),$$

$$L^{-1}\left[\frac{P_{1m}(v, s)}{P(v, s)}\right] = \sum_{j=1}^{N+2} A_{1m}^{(j)}(v) \exp(s_j(v) \tau), \quad L^{-1}\left[\frac{P_{q+1,m}(v, s)}{Q_q(v, s)}\right] = \sum_{j=1}^{N+3} A_{q+1,m}^{(j)}(v) \exp(s_j(v) \tau), \quad (18)$$

$$A_{km}^{(j)}(v) = \frac{P_{km}(v, s_j)}{P'(v, s_j)}, \quad A_{q+1,m}^{(j)}(v) = \frac{P_{q+1,m}(v, s_j)}{Q'_q(v, s_j)},$$

where $s_k = s_k(v) (k = \overline{1, N+2})$ - zeros of polynomial $P(v, s)$, $s_{N+3}(v) = -D_q v^2$, $H(\tau)$ - Heaviside function, L^{-1} - inverse operator of Laplace transform.

Inverse Fourier transform, sine and cosine transformation are done numerically through an algorithm presented in [12, 13].

Applying the Laplace transform and the Fourier series expansion (12) to system (8) for the mixed boundary-value problem (1) with homogeneous boundary conditions (6) leads to the system of linear algebraic equations (11). Its solution is (16), where $\lambda_n = \pi(2n-1)/2$. Accordingly, the transition to the originals space can be done using formulas (18).

4. Example

As a calculation example let us consider a problem for a two-component layer ($N=2$) with homogeneous boundary conditions (4). For body forces it is assumed:

$$F_1(x, \tau) = f_1(x)H(\tau), \quad F_2(x, \tau) = F_3(x, \tau) = 0, \quad f_1(x) = 1 - x.$$

The layer material is aluminum with the following properties [14]:

$$\lambda = 6.93 \cdot 10^{10} \frac{N}{m^2}, \quad \mu = 2.56 \cdot 10^{10} \frac{N}{m^2}, \quad T_0 = 800 \text{ K}, \quad \rho = 2700 \frac{kg}{m^3}, \quad L = 10^{-3} \text{ m},$$

$$\alpha_1 = 1.55 \cdot 10^7 \frac{J}{m^3}, \quad \alpha_2 = 6.14 \cdot 10^7 \frac{J}{m^3}, \quad D_1 = 7.73 \cdot 10^{-14} \frac{m^2}{s}, \quad D_2 = 3.11 \cdot 10^{-18} \frac{m^2}{s},$$

$$n_0^{(1)} = 0.95, \quad n_0^{(2)} = 0.05, \quad m^{(1)} = 0.027 \frac{kg}{mol}, \quad m^{(2)} = 0.064 \frac{kg}{mol}.$$

The calculation of convolutions (9) with considering (12) and (18) will result into ($q=1, 2$):

$$u = 2 \sum_{n=1}^{\infty} \sum_{l=1}^4 A_{1l}^{(l)}(\lambda_n) \left[\exp(s_l(\lambda_n) \tau) - 1 \right] \frac{\sin \lambda_n x}{\lambda_n s_l(\lambda_n)},$$

$$\eta_q = 2 \sum_{n=1}^{\infty} \sum_{l=1}^5 A_{q+1,l}^{(l)}(\lambda_n) \left[\exp(s_l(\lambda_n) \tau) - 1 \right] \frac{\cos \lambda_n x}{\lambda_n s_l(\lambda_n)}.$$

The results of calculations are represented at the graphics below. Solid line is $x=0.1$, dotted is $x=0.3$, dashed is $x=0.5$.

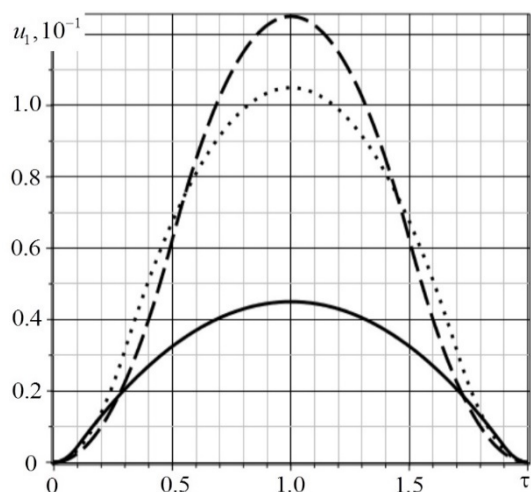


Fig. 1. The dependence of $u(x, \tau)$ on τ

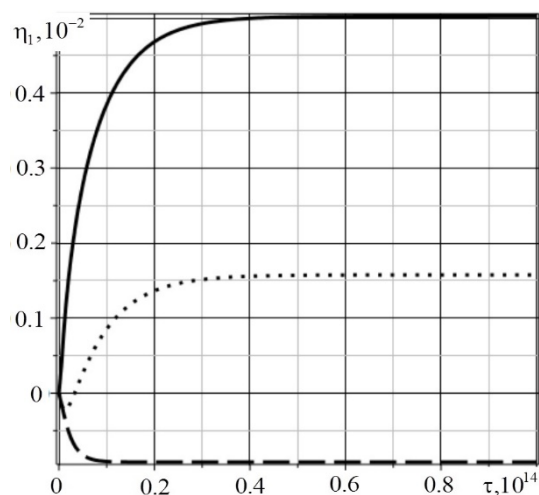


Fig. 2. The dependence of $\eta_1(x, \tau)$ on τ

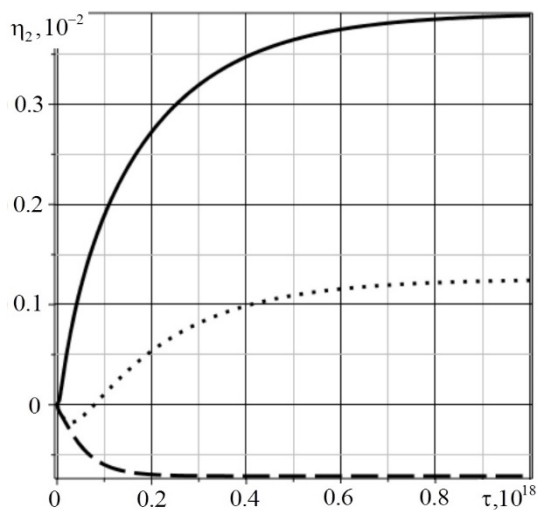


Fig. 3. The dependence of $\eta_2(x, \tau)$ on τ

This result is in addition to the results obtained earlier in [11-13,15] when were considered problems of constructing surface Green's functions.

5. Conclusions

The proposed algorithm for finding Green's function allows one to determine the fields of displacements and the concentrations increments of the medium components for given bulk perturbations. The numerical-analytical method used in the article made it possible to obtain exact solutions in the explicit form. To demonstrate the algorithm operation, an example is considered for a two-component medium (duralumin) which illustrating the interaction effects of mechanical and diffusion fields.

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